Econometrics - Heteroscedasticity

Master 1 Semestre 2 - EPOLPRO (IEDES)

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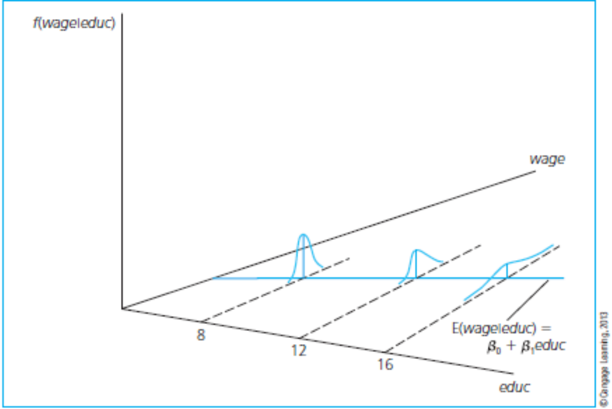
Consequences of heteroscedasticity on OLS estimates

The problem

FIGURE 2.9 Var(wage|educ) increasing with educ.



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- If u_i are heteroscedastic, then $V(\widehat{\beta_1}) \neq$ constante $(\neq \frac{\sigma^2}{SCT_x})$

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- $Var(\widehat{\beta_1})$: it is biased (OLS no more efficient)
- and thus, on the validity of inferences made (based on IC, t-stat, F stat, etc.)

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- Can we test the presence of heteroscedasticity ?

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- Can we test the presence of heteroscedasticity ?
- Is there a way to estimate $V(\widehat{\beta}_1)$ that is minimized and determine if t and F follow known distributions ?

Testing the presence of heteroscedasticity

The Breusch-Pagan test (1)

Starting point

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- Pb: We do not know u^2 !

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  - This last test is known as the Breusch Pagan Test

The Breusch-Pagan test (4)

Steps for implementing the Breusch-Pagan test

1. Estimate (*) and save the residuals
2. Estimate (**) and save the value of the R2
3. Compute the value of F or of LM
4. Conclude at a given significance level
5. If H0 is rejected: heteroscedasticity cannot be ignored and corrections should be applied

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- The White test is based on this result

The White test (2)

Let's write

- $\widehat{u^2} = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_1^2 + \delta_4 x_2^2 + \delta_5 x_1 x_2 + error$ $(k = 2)$ (iv)

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- H0 : all δ_j (except the constant) equal zero
- We know how to implement this test
- Drawbacks ?
  - If at a start $k = 2$, # of paramters to estimate for the test amounts to 5; If at a start $k = 3$, # of paramters to estimate for the test amounts to 9; If at a start $k = 6$, # of paramters to estimate for the test amounts to 27;

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  - $\widehat{u^2} = \gamma_0 + \gamma_1 \widehat{y} + \gamma_2 \widehat{y}^2 + error$ (v)

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Steps for implementing the White test

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3. Compute the value of F (or of LM)
4. Conclude at a given significance level
5. If H0 is rejected: heteroscedasticity cannot be ignored and corrections should be applied

Let's suppose H0 is rejected. What do we do then ?

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- Option 2: We estimate the model using weighted least square
- Both methods have drawbacks and advantages !

OLS inference robust to heteroscedasticity

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- u are heteroscedastic
  - Then, $V(\widehat{\beta_1})$ does not simplify to equal $\frac{\sigma^2}{SCT_x}$
  - $V(\widehat{\beta_1}) = \frac{\Sigma(SCT_x * V[u/x])}{SCT_x^2}$

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- u are heteroscedastic
  - Then, $V(\widehat{\beta_1})$ does not simplify to equal $\dfrac{\sigma^2}{SCT_x}$
  - $V(\widehat{\beta_1}) = \dfrac{\Sigma(SCT_x * V[u/x])}{SCT_x^2}$
  - How estimate $V(\widehat{\beta_1})$?
- White (1980) suggests to measure $\widehat{V(\widehat{\beta_1})}$ with $\dfrac{\Sigma(SCT_x * \widehat{u}^2)}{SCT_x^2}$ {(not demonstrated – for a discussion see p396 in Wooldridge (2013))}.

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- [$\rightarrow$] If n large: report systematically the robust standard errors and perform the usual tests, as usual (the command is automatized in most statistical softwares)
- [$\rightarrow$] If n *NOT* large, prefer alternative methods

Weighted least square estimation

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- We assume (A1)' to (A4)' are verified
- We also assume that $V(u/X) = E(u^2/X) = E(u^2) = \sigma^2 * h(X)$ with $h(x) > 0$ (since V()>0) ($h(X)$ is assumed to be known)

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- Let's rewritte (*) : $\frac{y}{\sqrt{h}} = \frac{\beta_0}{\sqrt{h}} + \frac{\beta_1}{\sqrt{h}}x_1 + \frac{\beta_2}{\sqrt{h}}x_2 + (...) + \frac{u}{\sqrt{h}}$ (++)

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- The hypothesis of the CLM being verified in $(++)$, we can estimate it in OLS and obtain unbiased $\widehat{\beta}_j^*$ (the coefficient in $(++)$) and $V(\widehat{\beta}_j^*)$
- Recall $V(\widehat{\beta}_j)$ are biased $\rightarrow$ prefer estimating $(++)$

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  - Apply the white correction method on (++) ?